Review

Directions: Do the problems. No calculators.

1. Find each limit:

a. $\lim\limits_{x\to 3} \frac{x^2-9}{x^3-8x^2+15x}$

b. $\lim\limits_{x\to 2} \frac{x^2-7x+6}{x^2-4}$

c. $\lim\limits_{x\to 8} \frac{x^2-9x+8}{x^2-16}$

d. $\lim\limits_{x\to 0} \frac{3\sin 7x}{2x}$

e. $\lim\limits_{x\to 0} \frac{\sin^2 5x}{2x^2}$

f. $\lim\limits_{x\to 0} \frac{\sin^5 2x}{3x^5}$

g. $\lim\limits_{x\to 0} \frac{2x-3\sin x}{4x-2}$

h. $\lim\limits_{x\to 0} \frac{3x^3-2\sin 4x}{x^2-x}$

i. $\lim\limits_{x\to 0} \frac{2\sin^3 3x}{4x^3}$

j. $\lim\limits_{x\to \infty} \frac{7x^3}{e^x}$

k. $\lim\limits_{x\to \infty} \frac{\sqrt{x}}{\ln x}$

l. $\lim\limits_{x\to \infty} \frac{2^x}{x^{100}}$

m. $\lim\limits_{x\to \infty} \frac{\sqrt[3]{x}}{x}$

n. $\lim\limits_{x\to \infty} e^{-x^2+x}$

o. $\lim\limits_{x\to -\infty} e^{-x^3-x-1}$

p. $\lim\limits_{x\to \infty} \ln(x^2-1)$

q. $\lim\limits_{x\to -\infty} \ln(x^2+x+1)$

r. $\lim_{x\to-\infty} \ln(x^3 - 5x + 7)$

s. $\lim_{x\to\infty} \ln(e^x - x^3)$

2. Find the values of a and b that make the following piecewise functions continuous for all real numbers. Use the definition of continuity in your answer.

a. $$f(x) = \begin{cases} x^2 - x - 1 & \text{if } x < -2 \\ ax^2 + bx & \text{if } -2 \le x \le 1 \\ \ln(x) + 5x - 2 & \text{if } x > 1 \end{cases}$$

b. $$f(x) = \begin{cases} 3x - 8 & \text{if } x < -2 \\ ax^2 + bx & \text{if } -2 \le x \le 1 \\ x^2 - x - 1 & \text{if } x > 1 \end{cases}$$

c. $$f(x) = \begin{cases} \frac{1}{x} + 2x & \text{if } x < -1 \\ ax^2 + bx & \text{if } -1 \le x < 2 \\ 3x - x^3 & \text{if } x \ge 2 \end{cases}$$

d. $$f(x) = \begin{cases} 3x - x^3 & \text{if } x < -1 \\ ax^2 + bx & \text{if } -1 \le x < 2 \\ \sqrt{2x} - x + 5 & \text{if } x \ge 2 \end{cases}$$

3. Find and classify all discontinuities of the following functions:

a. $f(x) = \frac{x^2-4x+3}{x^2-1}$

b. $f(x) = \frac{x^2-4}{x^3-7x^2+10x}$

c. $f(x) = \frac{x^3-9x}{x^3+2x^2-15x}$

4. Find the derivative of each of the following functions:

a. $y = \sin(e^x)$

b. $y = e^{\cos x}$

c. $y = (10x - 3)^6$

d. $y = (\cos x)^x$

e. $y = \frac{x}{\tan x}$

f. $y = (\tan^{-1} x)(3^x)$

g. $y = x^{(\ln x)}$

h. $y = \sqrt[3]{3x - 6}$

i. $y = \sqrt{\sin^{-1} x^2}$

j. $y = x^2 \sec^{-1} x$

k. $y = \log_2(\sec x)$

l. $y = (3x)^{\cot x}$

m. $y = (x^3 - 1)^\pi$

5. Find the instantaneous rate of change of each of the following functions at the given x-value.

a. $y = \sqrt{x^2 - x - 2}$ at $x = 3$

b. $y = \sin 3x$ at $x = \frac{\pi}{4}$

c. $y = (2x - 8)^3$ at $x = 5$

d. $y = \ln(x^2 - 3x)$ at $x = -1$

e. $y = e^{3x}$ at $x = \ln 2$

6. Find the slope of the tangent line to each of the following functions at the given x-value.

a. $y = (2x^2 - 3x + 1)^4$ at $x = 2$

b. $y = \sin^{-1}(x^2)$ at $x = \frac{1}{2}$

c. $y = \sqrt[3]{x^2 + 4}$ at $x = 2$

d. $y = \frac{1}{x^2 - 8x}$ at $x = 1$

e. $y = \ln(x^2 - x + 2)$ at $x = 3$

7. Find the equation of the tangent line and normal line to each of the following functions at each of the given x-values.

a. $y = \cos^3 x$ at $x = \frac{\pi}{3}$

b. $y = \sin x \cos x$ at $x = \frac{\pi}{4}$

c. $y = \sqrt{x^3 - 2}$ at $x = 3$

d. $y = \left(2x - \frac{1}{x}\right)^3$ at $x = 1$

e. $y = \ln(x^2 - 1)$ at $x = 3$

8. Approximate the change in the surface area of a sphere if the radius changes from 2 cm to 2.2 cm. The surface area of a sphere is $A = 4\pi r^2$.

9. Approximate the change in the volume of a sphere if the radius changes from 3 cm to 3.1 cm. The volume of a sphere is $V = \frac{4}{3}\pi r^3$.

10. Approximate the change in the area of an equilateral triangle if the side lengths all increase from 5 inches to 5.1 inches. The area of an equilateral triangle is $A = \frac{s^2\sqrt{3}}{4}$.

11. Approximate the change in the volume of a cylinder if the radius changes from 1 foot to 1.1 feet and the height changes from 3 feet to 2.9 feet. The volume of a cylinder is $V = \pi r^2 h$.

12. Approximate the change in the volume of a cone if the height stays constant at 2 inches but the radius decreases from 3 inches to 2.9 inches. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.

13. Approximate the value of $f(1.1)$ for $f(x) = x^3 - 2x + 6$ by using the tangent line at $x = 1$. Is your approximation an underestimate or an overestimate? Why?

14. Approximate the value of $f(4.9)$ for $f(x) = \frac{x+2}{x-3}$ by using the tangent line at $x = 5$. Is your approximation an underestimate or an overestimate?

15. Approximate the value of $f(3.1)$ for $f(x) = (2x + 1)^5$ by using the tangent line at $x = 3$. Is your approximation and underestimate or an overestimate?

16. Approximate the value of $f(2)$ for $f(x) = x^4 - 2x^2 - x - 1$ by using the secant line on the interval $[1,3]$. Is your approximation an underestimate or an overestimate?

17. Approximate the value of $f(0.5)$ for $f(x) = (3x - 2)^4$ by using the secant line on the interval $[0,1]$. Is your approximation an underestimate or an overestimate?

18. Approximate the value of $f(-1)$ for $f(x) = \frac{x-3}{x-2}$ by using the secant line on the interval $[-2,0]$. Is your approximation an underestimate or an overestimate?

19. Use linearization to approximate $\sqrt{102}$.

20. Use linearization to approximate $\sqrt[3]{9}$.

21. Use linearization to approximate $\sqrt{62}$.

22. Use linearization to approximate $\sqrt[3]{25}$.

23. Use linearization to approximate $\sqrt[4]{15}$.

24. Find all intervals on which the following functions are increasing and decreasing. Determine all local and absolute maxima and minima. Find all intervals on which the functions are concave up and concave down. Find all points of inflection.

a. $f(x) = x^4 - 18x^2 + 4$

b. $f(x) = x^3 - 6x^2 + 9x + 4$

c. $f(x) = 2x^3 + 21x^2 - 108x + 6$

d. $f(x) = x^4 - 6x^2 + 3$

e. $f(x) = 2xe^x$

25. Find all values that satisfy the Mean Value Theorem, if applicable, for each of the following functions on the given intervals:

a. $f(x) = \frac{1}{x}$ on $[1,2]$

b. $f(x) = \frac{5}{x-1}$ on $[0,2]$

c. $f(x) = \frac{2}{4-x}$ on $[0,2]$

d. $f(x) = x^3 - 3x + 1$ on $[1,2]$

e. $f(x) = \sqrt{x+2}$ on $[-1,2]$

26. What point on the graph of $y = \sqrt{x}$ is closest to the point $(3,0)$?

27. What point on the graph of $f(x) = x^{\frac{3}{2}} + 1$ is closest to the point $(2,1)$?

28. Sand is falling into a conical pile in which the radius is always four times the height. If sand is falling at a rate of 16 cubic inches per second, at what rate is the radius of the pile changing when the radius is 12 inches?

29. Sand is falling into a conical pile in which the height is always three times the radius. If the height of the pile is increasing at a rate of 12 inches per second, at what rate is the volume of the pile changing when the height is 20 inches?

30. Sand is falling into a conical pile in which the radius is always twice the height. If sand is falling at a rate of 40 cubic inches per second, at what rate is the radius of the pile changing when the height is 30 inches?

31. A 25 foot long ladder is leaning against a wall that meets the ground at a right angle. The ladder begins to slide down the wall. At the instant when the top of the ladder is 7 feet above the ground, the top of the ladder is sliding down at 10 feet per second. Find each of the following at that same instant:

a. The rate of change of the bottom of the ladder.

b. The rate of change of the area of the triangle formed by the floor, wall, and ladder.

c. The rate of change of the angle formed between the ladder and the floor.

32. A 17 foot long piece of wood is leaning against a wall that meets the ground at a right angle. The wood begins to slide down the wall. At the instant when the bottom of the piece of wood is 15 feet from the base of the wall, the bottom of the piece of wood is sliding at 5 feet per second. Find each of the following at that same instant:

a. The rate of change of the top of the piece of wood.

b. The rate of change of the area of the triangle formed by the floor, wall, and the piece of wood.

c. The rate of change of the angle formed between the piece of wood and the floor.

33. A 13 foot long ladder is leaning against a wall that meets the ground at a right angle. The ladder begins to slide down the wall. At the instant when the top of the ladder is 12 feet above the ground, the bottom of the ladder is sliding away from the wall at 2 feet per second. Find each of the following at that same instant:

a. The rate of change of the top of the ladder.

b. The rate of change of the area of the triangle formed by the floor, wall, and ladder.

c. The rate of change of the angle formed between the ladder and the floor.

34. A rectangle is inscribed underneath the parabola $y = 4 - x^2$. How far from the origin should the bottom right vertex of the rectangle be so that the area of the rectangle is at a maximum?

35. A rectangle is inscribed underneath the parabola $y = 6x - x^2$. What length (the side stretching across the x-axis) maximizes the area of the rectangle?

36. A rectangle is inscribed underneath the semicircle $y = \sqrt{4 - x^2}$. How far from the origin should the bottom right vertex of the rectangle be so that the area of the rectangle is at a maximum?