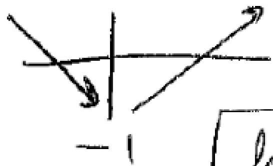
1. Determine all local extrema of f, all intervals on which f is increasing or decreasing, all intervals of concavity, and all points of inflection.

a. $y = xe^x$

$y' = xe^x + e^x = e^x(x+1)$

y' undefined never $\quad y' = 0$ if $x+1=0 \rightarrow x=-1$

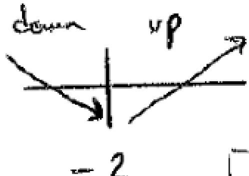


dec: $(-\infty, -1]$ inc: $[-1, \infty)$ $\quad f(-1) = (-1)(e^{-1}) = -\frac{1}{e}$

local min of $-\frac{1}{e}$ at $x = -1$

$y'' = xe^x + e^x + e^x = e^x(x+1+1) = e^x(x+2)$ $\quad y''$ undefined never

$y'' = 0$ if $x+2=0$, $x=-2$



$f(-2) = -2(e^{-2}) = -\frac{2}{e^2}$

concave down: $(-\infty, -2)$ concave up: $(-2, \infty)$ POI: $\left(-2, -\frac{2}{e^2}\right)$

b. $y = x\sqrt{8 - x^2}$

$$y' = \frac{x(-2x)}{2\sqrt{8-x^2}} + \sqrt{8-x^2} = \frac{-x^2 + 8 - x^2}{\sqrt{8-x^2}} = \frac{8-2x^2}{\sqrt{8-x^2}}$$

$y' =$ if $8-2x^2=0$

$x = \pm 2$

y' undefined if $8-x^2=0$

$x = \pm 2\sqrt{2}$

$-2\sqrt{2}$ $\quad -2 \quad$ 2 $\quad 2\sqrt{2}$

$f(-2\sqrt{2}) = 0$

$f(2\sqrt{2}) = 0$

$f(-2) = -4$

$f(2) = 4$

$$y'' = \frac{-4x\sqrt{8-x^2} - \frac{(8-2x^2)(-\cancel{2}x)}{\cancel{2}\sqrt{8-x^2}}}{8-x^2}$$

$$= \frac{-4x(8-x^2) + 8x - 2x^3}{\sqrt{8-x^2}\,(8-x^2)} = \frac{-24x + 2x^3}{\sqrt{8-x^2}\,(8-x^2)}$$

$y'' = 0$ if $2x^3 - 24x = 0 \rightarrow 2x(x^2-12) = 0$

$x = \pm 2\sqrt{3}$ $\quad x = 0$

y'' undefined if $x = \pm 2\sqrt{2} \rightarrow$ endpoints

not in domain

0

$f(0) = 0$

dec: $[-2\sqrt{2}, -2] \cup [2, 2\sqrt{2}]$

inc: $[-2, 2]$

conc. up: $(-2\sqrt{2}, 0)$

conc. down: $(0, 2\sqrt{2})$

local min of -4 at $x = -2$

local min of 0 at $x = 2\sqrt{2}$

local max of 0 at $x = -2\sqrt{2}$

local max of 4 at $x = 2$

POI: $(0,0)$

c. $y = x\sqrt[3]{3x+2}$

$$y' = \frac{\cancel{3}x}{\cancel{3}(3x+2)^{\frac{2}{3}}} + \sqrt[3]{3x+2}$$

$$= \frac{x + 3x+2}{(3x+2)^{2/3}} = \frac{4x+2}{(3x+2)^{2/3}}$$

$$y'' = \frac{4(3x+2)^{2/3} - \frac{(4x+2)(2)(\cancel{3})}{\cancel{3}(3x+2)^{1/3}}}{(3x+2)^{4/3}}$$

$$= \frac{4(3x+2) - 8x - 4}{(3x+2)^{5/3}} = \frac{4x+4}{(3x+2)^{5/3}}$$

$y' = 0$ if $x = -\frac{1}{2}$

y' undefined if $x = -\frac{2}{3}$

$\searrow \; | \; \searrow \; | \; \nearrow$

$-\frac{2}{3} \quad -\frac{1}{2}$

$f(-1) = 1$

$f(-\frac{2}{3}) = 0$

$f(-\frac{1}{2}) = -\frac{1}{2\sqrt[3]{2}}$

$y'' = 0$ if $x = -1$

y'' undefined if $x = -\frac{2}{3}$

up | down | up

$-1 \quad -4/3$

Inc: $[-\frac{1}{2}, \infty)$

dec: $(-\infty, -\frac{1}{2}]$

local min of $-\frac{1}{2\sqrt[3]{2}}$ at $x = -\frac{1}{2}$

concave up: $(-\infty, -1) \cup (-\frac{2}{3}, \infty)$

concave down: $(-1, -\frac{2}{3})$

POI: $(-1, 1)$

$(-\frac{2}{3}, 0)$

d. $f(x) = x^3 + 10x^2 + 25x - 50$

$f'(x) = 3x^2 + 20x + 25$

$f' = 0$ if $3x^2 + 20x + 25 = 0$

$(3x+5)(x+5) = 0$

$x = -\frac{5}{3}$ $\quad x = -5$

$f''(x) = 6x + 20$

$f'' = 0$ if $x = -\frac{10}{3}$

f': $+$ | $-$ | $+$ at -5 and $-5/3$

f'': down | up at $-10/3$

$f(-5) = -50$

$f(-\frac{5}{3}) = -\frac{1850}{27}$

$f(-\frac{10}{3}) = -\frac{1600}{27}$

inc: $(-\infty, -5] \cup [-\frac{5}{3}, \infty)$

dec: $[-5, -\frac{5}{3}]$

conc. up: $(-\frac{10}{3}, \infty)$

conc. down: $(-\infty, -\frac{10}{3})$

local max of -50 at $x = -5$

local min of $\frac{-1850}{27}$ at $x = -\frac{5}{3}$

POI: $(-\frac{10}{3}, -\frac{1600}{27})$

e. $f(x) = (x^2 - 1)^2$

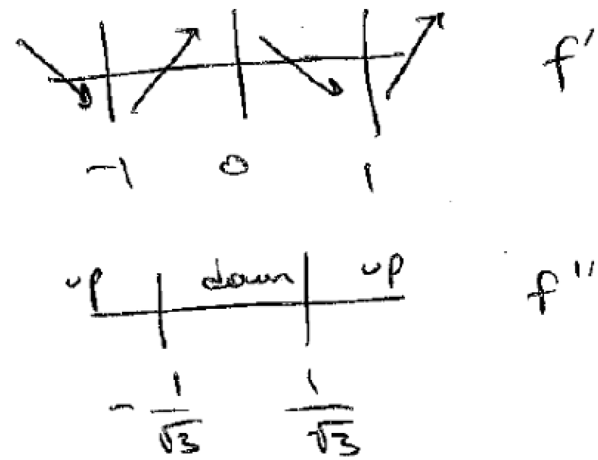
$f'(x) = 2(2x)(x^2-1) = 4x(x^2-1) = 4x^3 - 4x$

$f' = 0$ if $x = 0$ $x = \pm 1$

$f''(x) = 12x^2 - 4$

$f'' = 0$ if $x^2 = \frac{1}{3}$

$x = \pm \frac{1}{\sqrt{3}}$



$f(-1) = 0$

$f(1) = 0$

$f(0) = 1$

$f\left(-\frac{1}{\sqrt{3}}\right) = \frac{4}{9}$ $f\left(\frac{1}{\sqrt{3}}\right) = \frac{4}{9}$

inc: $[-1, 0] \cup [1, \infty)$ dec: $(-\infty, -1] \cup [0, 1]$

conc up: $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$

conc. down: $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

local min of 0 at $x = -1$, $x = 1$ local max of 1 at $x = 0$

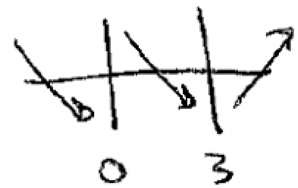
POI: $\left(-\frac{1}{\sqrt{3}}, \frac{4}{9}\right)$ $\left(\frac{1}{\sqrt{3}}, \frac{4}{9}\right)$

f. $f(x) = x^4 - 4x^3 + 2$

$f' = 4x^3 - 12x^2$

$f' = 0 = 4x^3 - 12x^2 \to 4x^2(x-3) = 0$

$x = 0 \quad x = 3$



Inc: $[3, \infty)$

dec: $(-\infty, 3]$

conc up: $(-\infty, 0) \cup (2, \infty)$

conc. down: $(0, 2)$

local min of -25 at $x = 3$

POI: $(0, 2)$ $(2, -14)$

$f'' = 12x^2 - 24x = 0$

$12x(x-2) = 0$

$x = 0 \quad x = 2$

up | down | up

0 2

$f(0) = 2$

$f(2) = -14$

$f(3) = -25$

2. Sketch the graph of $f(x) = x^4 - 2x^3$ by finding all local extrema, intervals on which f is increasing and decreasing, asymptotes, intercepts, intervals of concavity, and all points of inflection.

$$f(x) = x^4 - 2x^3 = x^3(x-2) = 0$$

$x = 0 \quad x = 2 \rightarrow$ x-ints

$$f'(x) = 4x^3 - 6x^2 = 0 \text{ if } 2x^2(2x-3)$$

$x = 0 \quad x = \frac{3}{2}$

$0 \quad \frac{3}{2}$

$$f''(x) = 12x^2 - 12x = 0 \text{ if } 12x(x-1) = 0$$

$x = 0 \quad x = 1$

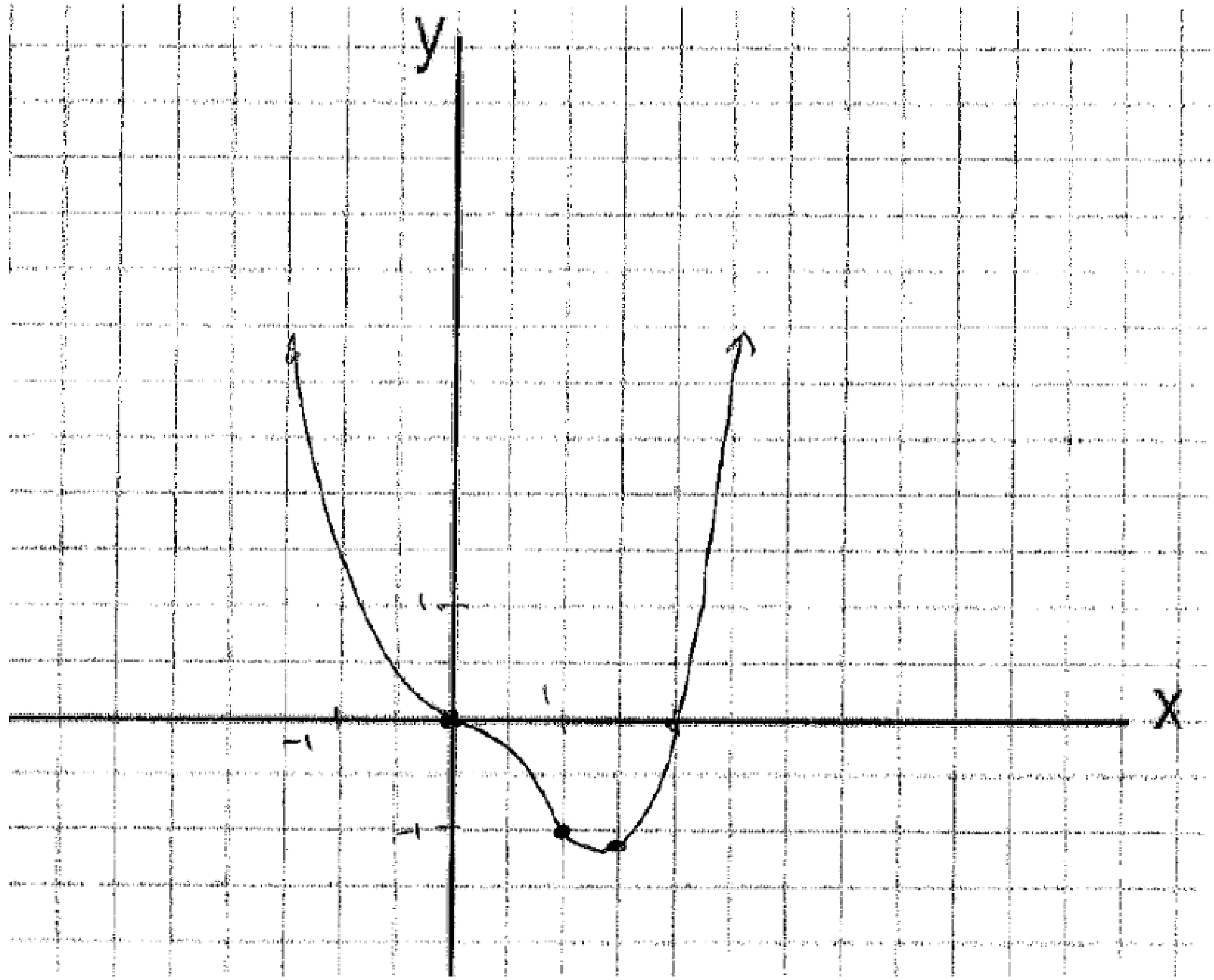
~~down~~ up | down | up

$0 \quad 1$

dec: $(-\infty, \frac{3}{2}]$ inc: $[3/2, \infty)$ local min of $-\frac{27}{16}$ at $x = \frac{3}{2}$

Conc. up: $(-\infty, 0) \cup (1, \infty)$ POI: $(0,0)$ $(1,-1)$

Conc. down: $(0,1)$


y
X
1
-1
1
-1

3. Sketch the graph of $y = \frac{-3x}{\sqrt{x^2+4}}$ by finding all local extrema, intervals on which f is increasing and decreasing, asymptotes, intercepts, intervals of concavity, and all points of inflection.

$$y = \frac{-3x}{\sqrt{x^2+4}}$$

x-ints $-3x=0 \rightarrow x=0$ $(0,0)$

$$y' = \frac{-3\sqrt{x^2+4} - \frac{(-3x)(\cancel{2}x)}{\cancel{2}\sqrt{x^2+4}}}{x^2+4} = \frac{-3(x^2+4)+3x^2}{(x^2+4)\sqrt{x^2+4}} = \frac{-12}{(x^2+4)^{3/2}}$$

dec: $(-\infty, \infty)$

no CV

$$y'' = \frac{\left[(x^2+4)\sqrt{x^2+4}\right]0 - \frac{(-12)3(2x)\sqrt{x^2+4}}{2}}{(x^2+4)^3} = \frac{72x}{2(x^2+4)^{5/2}} = \frac{36x}{(x^2+4)^{5/2}}$$

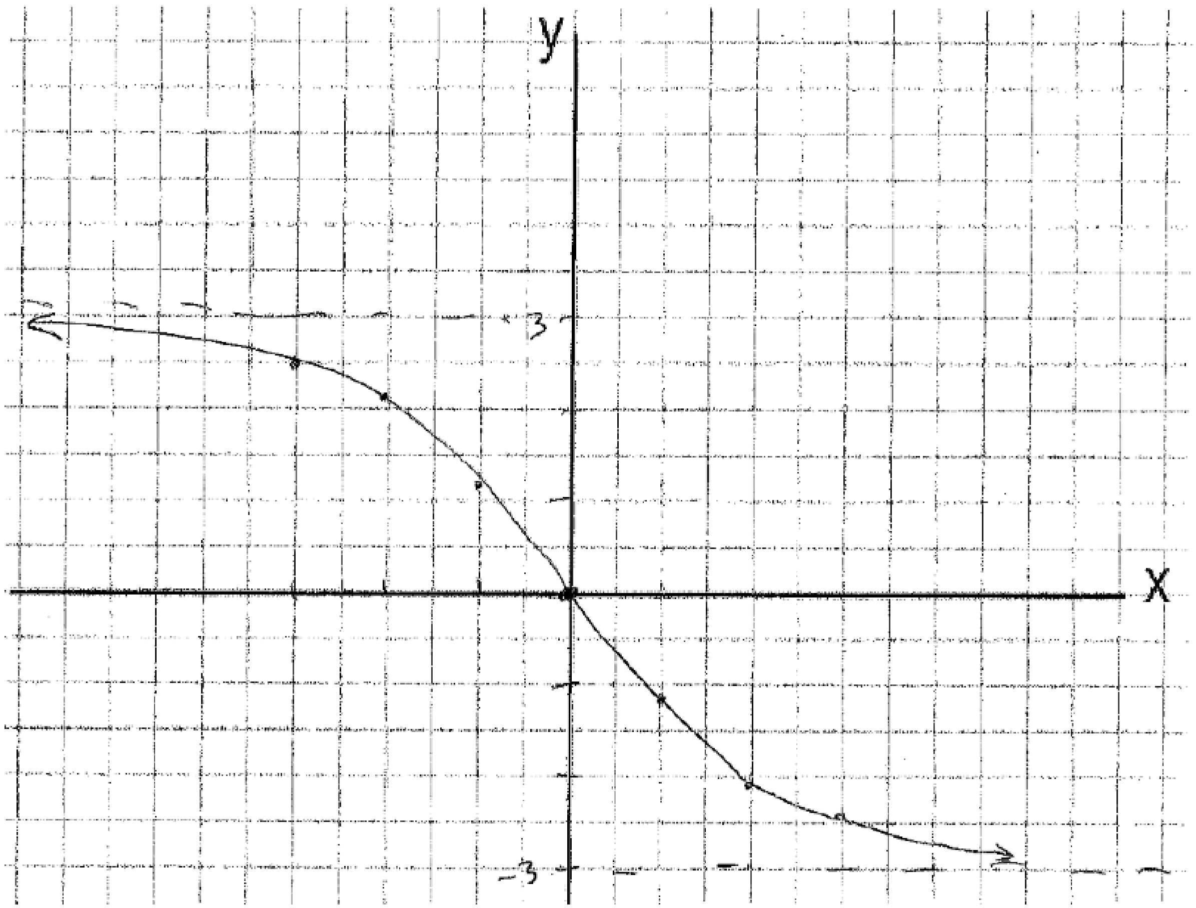
$x=0$

down | up

0

$(0,0)$

$\lim_{x \to \infty} \frac{-3x}{\sqrt{x^2+4}} = -3$ $y=-3$ is HA to right $\lim_{x \to -\infty} \frac{-3x}{\sqrt{x^2+4}} = 3$ $y=3$ is HA to left



4. Sketch the graph of $f(x) = \frac{4-x^2}{x+3}$ by finding all local extrema, intervals on which f is increasing and decreasing, asymptotes, intercepts, intervals of concavity, and all points of inflection.

$f(x) = \dfrac{4-x^2}{x+3}$ x-ints $4-x^2=0$

$x = \pm 2$ $(2,0)$ $(-2,0)$

VA: $x=-3$

$\lim_{x\to\infty} f(x) = -\infty$ no ~~HA~~

$\lim_{x\to-\infty} f(x) = \infty$ no HA

$$\begin{array}{r|rrr} -3 & -1 & 0 & 4 \\ & & 3 & -9 \\ \hline & -1 & 3 & -5 \end{array}$$

OA. $y = -x+3$

$$f'(x) = \frac{(x+3)(-2x)-(4-x^2)}{(x+3)^2} = \frac{-2x^2-6x-4+x^2}{(x+3)^2} = \frac{-x^2-6x-4}{(x+3)^2}$$

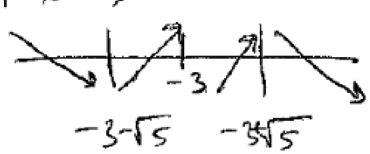
$= 0$ if $-x^2-6x-4=0$ $\quad \dfrac{6 \pm \sqrt{36-16}}{-2} = -3 \pm \dfrac{2\sqrt{5}}{-2} =$

undefined if $x=-3$

$-3 \pm \sqrt{5}$

$(-3+\sqrt{5}, \approx 1.528)$

$(-3-\sqrt{5}, \approx 10.472)$



$$f''(x) = \frac{(x^2+6x+9)(-2x-6) - (-x^2-6x-4)\,2(2x+6)}{(x+3)^4} = \frac{(2x+6)(-x^2-6x-9+x^2+6x+4)}{(x+3)^4}$$

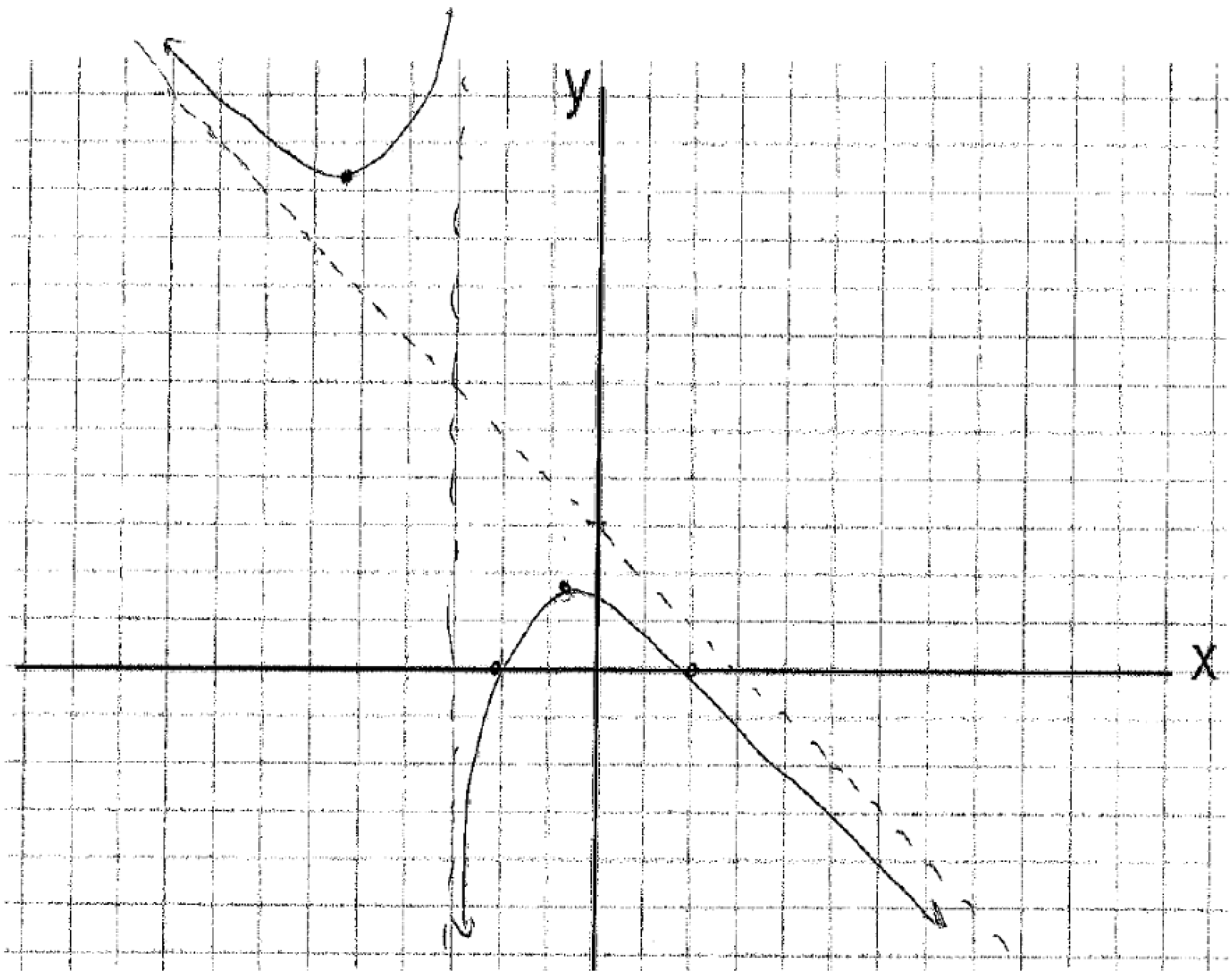
$$= \frac{2(-5)}{(x+3)^3} = \frac{-10}{(x+3)^3}$$

undefined at $x=-3$

up | down

-3

asymp



y
X

5. Sketch the graph of $y = \frac{(x+1)^2}{x^2+1}$ by finding all local extrema, intervals on which f is increasing and decreasing, asymptotes, intercepts, intervals of concavity, and all points of inflection.

$$y = \frac{(x+1)^2}{x^2+1} = \frac{x^2+2x+1}{x^2+1}$$

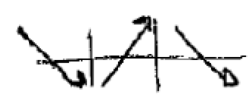
HA: $y=1$
no VA
no OA

x-int: $(-1,0)$
y-int: $(0,1)$

$$y' = \frac{(x^2+1)(2x+2)-(x^2+2x+1)(2x)}{(x^2+1)^2} = \frac{2x^3+2x^2+2x+2-2x^3-4x^2-2x}{(x^2+1)^2}$$

$$= \frac{-2x^2+2}{(x^2+1)^2}$$

$= 0$ if $-2x^2+2=0 \rightarrow x^2=1 \rightarrow x=\pm 1$
undefined never



$-1 \quad 1$

$f(-1)=0 \qquad f(1)=2$

$$y'' = \frac{(x^2+1)^2(-4x) - (-2x^2+2)(2x)(2)(x^2+1)}{(x^2+1)^4}$$

$$= \frac{-4x(x^4+2x^2+1) - (-2x^2+2)(4x^3+4x)}{(x^2+1)^4} = \frac{-4x^5 - 8x^3 - 4x - (-8x^5 - 8x^3 + 8x^3 + 8x)}{(x^2+1)^4}$$

$$= \frac{-4x^5 - 8x^3 - 4x + 8x^5 - 8x}{(x^2+1)^4} = \frac{4x^5 - 8x^3 - 12x}{(x^2+1)^4} = \frac{4x(x^4 - 2x^2 - 3)}{(x^2+1)^4}$$

$$= \frac{4x(x^2-3)(x^2+1)}{(x^2+1)^4} = \frac{4x(x^2-3)}{(x^2+1)^3}$$

$= 0$ if $x = 0$, $x = \pm\sqrt{3}$

undefined nowhere

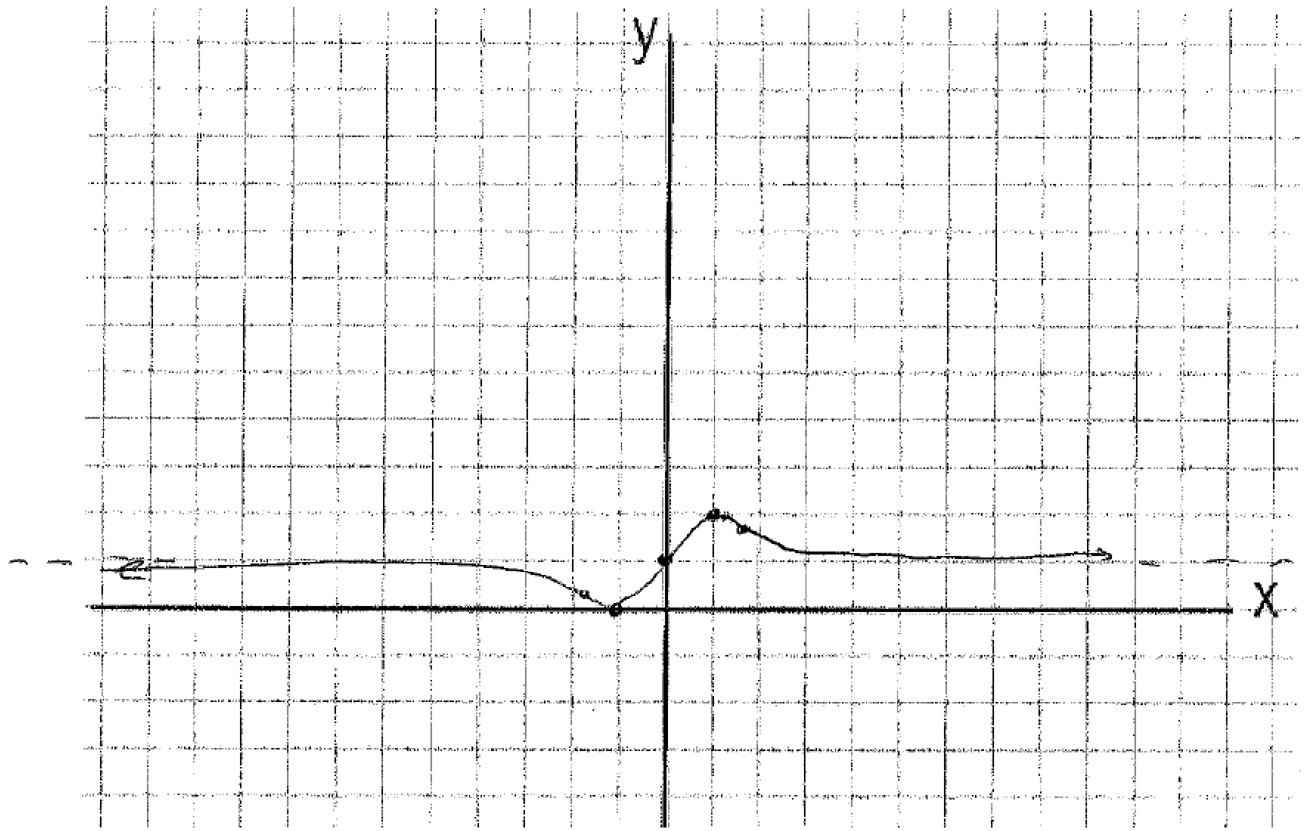
down | up | down | up

$-\sqrt{3}$ $\quad 0 \quad$ $\sqrt{3}$

$f(0) = 1$

$f(-\sqrt{3}) = \frac{(1-\sqrt{3})^2}{4}$

$f(-\sqrt{3}) = \frac{(1+\sqrt{3})^2}{4}$


y
X