Unit 1 Homework Packet

Directions: Complete each problem and show all work. Make sure to clearly box or circle your answers.

1. Find each limit (if it exists):

a. $\lim_{x\to 3}(x^2+2)$ $= 3^2+2 = \boxed{11}$

b. $\lim_{x\to 100} 7$ $= \boxed{7}$

c. $\lim_{x\to 5}\frac{x+2}{x-4}$ $= \frac{5+2}{5-4} = \frac{7}{1} = \boxed{7}$

d. $\lim_{x\to 2}\frac{x^2-4}{x-2}$ $= \lim_{x\to 2}\frac{(x+2)(x-2)}{(x-2)} = \lim_{x\to 2}(x+2) = 2+2 = \boxed{4}$

e. $\lim_{x\to 1}\frac{x^2-x}{2x^2+5x-7}$ $= \lim_{x\to 1}\frac{x(x-1)}{(x-1)(2x+7)} = \lim_{x\to 1}\frac{x}{2x+7} = \frac{1}{2(1)+7} = \boxed{\frac{1}{9}}$

f. $\lim_{z\to 4}\frac{z^2-16}{\sqrt{z}-2}$ $= \lim_{z\to 4}\frac{(z+4)(z-4)}{\sqrt{z}-2} = \lim_{z\to 4}\frac{(z+4)(\sqrt{z}+2)(\sqrt{z}-2)}{\sqrt{z}-2} = \lim_{z\to 4}(z+4)(\sqrt{z}+2)$

$= (4+4)(\sqrt{4}+2) = (8)(4) = \boxed{32}$

g. $\lim_{x\to 25}\frac{\sqrt{x}-5}{x-25}$ $= \lim_{x\to 25}\frac{\sqrt{x}-5}{(\sqrt{x}-5)(\sqrt{x}+5)} = \lim_{x\to 25}\frac{1}{\sqrt{x}+5} = \frac{1}{5+5} = \boxed{\frac{1}{10}}$

h. $\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$ $= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0}\frac{2xh+h^2}{h} = \lim_{h\to 0}\frac{h(2x+h)}{h}$

$= \lim_{h\to 0}(2x+h) = \boxed{2x}$

i. $\lim_{a\to -2}\frac{a-4}{a^2-2a-8}$ $= \lim_{a\to -2}\frac{a-4}{(a-4)(a+2)} = \lim_{a\to -2}\frac{1}{a+2} = \frac{1}{-2+2} = \frac{1}{0}$

$\boxed{\text{DNE}}$

2. Find each limit, if it exists: (1) $\lim_{x\to a^-} f(x)$, (2) $\lim_{x\to a^+} f(x)$, (3) $\lim_{x\to a} f(x)$

a. $f(x) = \frac{x+5}{|x+5|}$; $a = -5$

(1) $\lim_{x\to -5^-} \frac{x+5}{|x+5|} = \frac{-}{+} = \boxed{-1}$

(2) $\lim_{x\to -5^+} \frac{x+5}{|x+5|} = \frac{+}{+} = \boxed{1}$

(3) $\lim_{x\to -5} \frac{x+5}{|x+5|} \to$ DNE

b. $f(x) = \sqrt{x-3} + 2$; $a = 3$

(1) $\lim_{x\to 3^-} \left(\sqrt{x-3} + 2\right) = \sqrt{-} + 2 \to \boxed{\text{DNE}}$

(2) $\lim_{x\to 3^+} \left(\sqrt{x-3} + 2\right) = 0 + 2 = \boxed{2}$

(3) $\lim_{x\to 3} \left(\sqrt{x-3} + 2\right)$ $\boxed{\text{DNE}}$

c. $f(x) = \frac{|x-1|}{x-1}$; $a = 1$

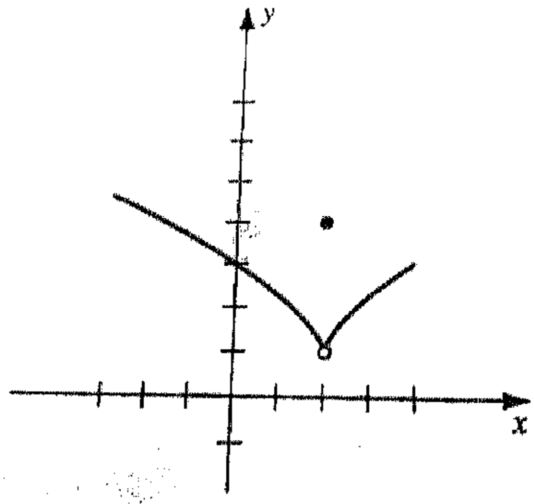
(1) $\lim_{x\to 1^-} \frac{|x-1|}{x-1} = \frac{+}{-} = \boxed{-1}$

(2) $\lim_{x\to 1^+} \frac{|x-1|}{x-1} = \frac{+}{+} = \boxed{1}$

(3) $\lim_{x\to 1} \frac{|x-1|}{x-1}$ $\boxed{\text{DNE}}$

3. Refer to the graph to find each limit (if it exists): (1) $\lim_{x \to 2^-} f(x)$, (2) $\lim_{x \to 2^+} f(x)$, (3) $\lim_{x \to 2} f(x)$, (4) $\lim_{x \to 0^-} f(x)$, (5) $\lim_{x \to 0^+} f(x)$, (6) $\lim_{x \to 0} f(x)$

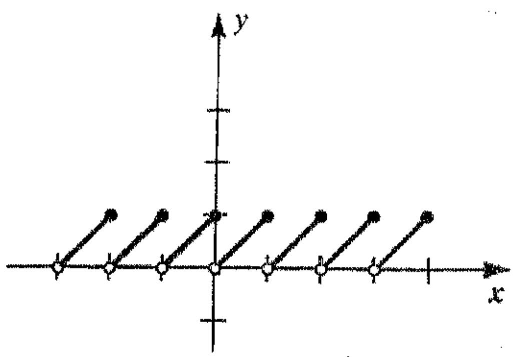
a.



(1) 1 (2) 1 (3) 1

(4) 3 (5) 3 (6) 3

b.



(1) 1 (2) 0 (3) DNE

(4) 1 (5) 0 (6) DNE

Use Squeeze Theorem to prove:

$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$ $\quad -1 \leq \sin\left(\frac{1}{x}\right) \leq 1$

Case 1: $x > 0$

$-x \leq x\sin\left(\frac{1}{x}\right) \leq x$

$\lim_{x\to 0^+} -x = 0$

$\lim_{x\to 0^+} x = 0$

$\therefore$, by Sandwich Theorem

$\lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right) = 0$

Case 2: $x < 0$

$-x \geq x\sin\left(\frac{1}{x}\right) \geq x$

$\lim_{x\to 0^-} -x = 0$

$\lim_{x\to 0^-} x = 0$

$\therefore$ by Sandwich Theorem

$\lim_{x\to 0^-} x\sin\left(\frac{1}{x}\right) = 0$

$\therefore \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

6. Find each of the following limits using properties of limits and already known limit values:

a. $\lim\limits_{x\to 0}\frac{x}{\sin(2x)}$

$$= \lim_{x\to 0}\frac{2x}{2\sin(2x)} = \frac{1}{2}\lim_{x\to 0}\frac{2x}{\sin(2x)} = \frac{1}{2}(1) = \boxed{\frac{1}{2}}$$

b. $\lim\limits_{\theta\to 0}\frac{3\theta+\sin(\theta)}{\theta}$

$$= \lim_{\theta\to 0}\left(\frac{3\theta}{\theta}\right) + \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \lim_{\theta\to 0} 3 + \lim_{\theta\to 0}\frac{\sin\theta}{\theta}$$

$$= 3 + 1 = \boxed{4}$$

c. $\lim\limits_{r\to 0}\frac{r\cos(r)-r^2}{r}$

$$= \lim_{r\to 0}\frac{r(\cos(r)-r)}{r} = \left[\lim_{r\to 0}\frac{r}{r}\right]\cdot\left[\lim_{r\to 0}\cos(r)-r\right]$$

$$= \left(\lim_{r\to 0} 1\right)\cdot\left[\lim_{r\to 0}\cos(r)-r\right] = 1\cdot\left[\cos(0)-0\right] = 1\cdot 1 = \boxed{1}$$

d. $\lim\limits_{x\to 0}\frac{\tan(10x)}{\sin(3x)}$

$$= \lim_{x\to 0}\frac{\frac{\sin(10x)}{\cos(10x)}}{\sin(3x)} = \lim_{x\to 0}\frac{\sin(10x)}{\sin(3x)\cos(10x)} = \left(\lim_{x\to 0}\frac{\sin(10x)}{\sin(3x)}\right)\left(\lim_{x\to 0}\frac{1}{\cos(10x)}\right)$$

$$= \left[\lim_{x\to 0}\frac{\sin(10x)}{1}\cdot\frac{1}{\sin(3x)}\right]\cdot\left[\frac{1}{\cos(0)}\right] = \left[\lim_{x\to 0}\frac{10\sin(10x)}{10x}\cdot\frac{3x}{3\sin(3x)}\right]\cdot 1$$

$$= \frac{10}{3}\left[\lim_{x\to 0}\frac{\sin(10x)}{10x}\cdot\frac{3x}{\sin(3x)}\right] = \frac{10}{3}\left\{\left[\lim_{x\to 0}\frac{\sin(10x)}{10x}\right]\cdot\left[\lim_{x\to 0}\frac{3x}{\sin 3x}\right]\right\} = \frac{10}{3}[1\cdot 1] = \boxed{\frac{10}{3}}$$

e. $\lim_{x\to 0}\frac{1+12x^2-2\cos(x)+\cos^2(x)}{3x^2}$ $= \lim_{x\to 0}\frac{12x^2}{3x^2} + \lim_{x\to 0}\frac{1-2\cos(x)+\cos^2(x)}{3x^2}$

$= \lim_{x\to 0} 4 + \lim_{x\to 0}\frac{\left[1-\cos(x)\right]^2}{3x^2} = 4 + \frac{1}{3}\left[\lim_{x\to 0}\left(\frac{1-\cos(x)}{x}\right)^2\right]$

$= 4 + \frac{1}{3}\left[\lim_{x\to 0}\frac{1-\cos(x)}{x}\right]^2 = 4 + \frac{1}{3}(0^2) = \boxed{4}$

f. $\lim_{z\to 0}\frac{\sin(7z)}{\sin(3z)}$ $= \lim_{z\to 0}\frac{\sin(7z)}{1}\cdot\frac{1}{\sin(3z)} = \lim_{z\to 0}\left[\frac{7\sin(7z)}{7z}\cdot\frac{3z}{3\sin(3z)}\right]$

$= \frac{7}{3}\left[\lim_{z\to 0}\frac{\sin(7z)}{7z}\cdot\frac{3z}{\sin(3z)}\right] = \frac{7}{3}\cdot\left\{\left[\lim_{z\to 0}\frac{\sin(7z)}{7z}\right]\cdot\left[\lim_{z\to 0}\frac{3z}{\sin(3z)}\right]\right\}$

$= \frac{7}{3}\left[1\cdot 1\right] = \boxed{\frac{7}{3}}$

7. Express each of the following limits as ∞, $-\infty$, or DNE (Does Not Exist): (1) $\lim_{x\to a^-} f(x)$, (2) $\lim_{x\to a^+} f(x)$, (3) $\lim_{x\to a} f(x)$

a. $f(x) = \frac{5}{x-4}$; $a = 4$

(1) $\lim_{x\to 4^-} f(x) = \frac{5}{\text{small} -} = \boxed{-\infty}$ (2) $\lim_{x\to 4^+} f(x) = \frac{5}{\text{small} +} = \boxed{\infty}$ (3) $\lim_{x\to 4} f(x)$ $\boxed{\text{DNE}}$

b. $f(x) = \frac{3x}{(x+8)^2}$; $a = -8$

(1) $\lim_{x\to -8^-} f(x) = \frac{-24}{\text{small} +} = \boxed{-\infty}$ (2) $\lim_{x\to -8^+} f(x) = \frac{-24}{\text{small} +} = \boxed{-\infty}$ $\lim_{x\to -\infty} f(x) = \boxed{-\infty}$

c. $f(x) = \frac{1}{x(x-3)^2}$; $a = 3$

(1) $\lim_{x \to 3^-} f(x) = \frac{1}{3(\text{small}+)} = \boxed{\infty}$ (2) $\lim_{x \to 3^+} f(x) = \frac{1}{3(\text{small}+)} = \boxed{\infty}$ (3) $\lim_{x \to 3} f(x) = \boxed{\infty}$

d. $f(x) = \frac{2x^2}{x^2-x-2}$; $a = -1$ $f(x) = \frac{2x^2}{(x-2)(x+1)}$

(1) $\lim_{x \to -1^-} f(x) = \frac{2}{(-3)(\text{small}-)} = \boxed{\infty}$ (2) $\lim_{x \to -1^+} f(x) = \frac{2}{(-3)(\text{small}+)} = \boxed{-\infty}$ (3) $\lim_{x \to -1} f(x)$ $\boxed{DNE}$

8. Find each limit if it exists:

a. $\lim\limits_{x\to\infty} \frac{5x^2-3x+1}{2x^2+4x-7} = \lim\limits_{x \to \infty} \frac{\frac{5x^2}{x^2} - \frac{3x}{x^2} + \frac{1}{x^2}}{\frac{2x^2}{x^2} + \frac{4x}{x^2} - \frac{7}{x^2}} = \lim\limits_{x \to \infty} \frac{5 - \frac{3}{x} + \frac{1}{x^2}}{2 + \frac{4}{x} - \frac{7}{x^2}}$

$= \frac{5-0+0}{2+0-0} = \boxed{\frac{5}{2}}$

b. $\lim\limits_{x\to\infty} \frac{2x^2-x+3}{x^3+1} = \lim\limits_{x \to \infty} \frac{\frac{2x^2}{x^3} - \frac{x}{x^3} + \frac{3}{x^3}}{\frac{x^3}{x^3} + \frac{1}{x^3}} = \lim\limits_{x \to \infty} \frac{\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x^3}}{1 + \frac{1}{x^3}}$

$= \frac{0-0+0}{1+0} = \frac{0}{1} = \boxed{0}$

c. $\lim\limits_{x\to-\infty} \frac{2-x^2}{x+3} = \lim\limits_{x \to -\infty} \frac{\frac{2}{x^2} - \frac{x^2}{x^2}}{\frac{x}{x^2} + \frac{3}{x^2}} = \lim\limits_{x \to -\infty} \frac{\frac{2}{x^2} - 1}{\frac{1}{x} + \frac{3}{x^2}} = \frac{0-1}{0+0} = \frac{-1}{0}$ $\boxed{DNE}$

d. $\lim\limits_{x\to-\infty} \frac{2x^2-3}{4x^3+5x} = \lim\limits_{x \to -\infty} \frac{\frac{2x^2}{x^3} - \frac{3}{x^3}}{\frac{4x^3}{x^3} + \frac{5x}{x^3}} = \lim\limits_{x \to -\infty} \frac{\frac{2}{x} - \frac{3}{x^3}}{4 + \frac{5}{x^2}} = \frac{0-0}{4+0} = \frac{0}{4} = \boxed{0}$

e. $\lim\limits_{x\to\infty} \sqrt[3]{\frac{8+x^2}{x(x+1)}} = \sqrt[3]{\lim\limits_{x \to \infty} \frac{8+x^2}{x^2+x}} = \sqrt[3]{\lim\limits_{x \to \infty} \frac{\frac{8}{x^2} + \frac{x^2}{x^2}}{\frac{x^2}{x^2} + \frac{x}{x^2}}} = \sqrt[3]{\lim\limits_{x \to \infty} \frac{\frac{8}{x^2} + 1}{1 + \frac{1}{x}}} = \sqrt[3]{\frac{0+1}{1+0}}$

$= \sqrt[3]{1} = \boxed{1}$

f. $\lim_{x\to-\infty} \frac{4x-3}{\sqrt{x^2+1}}$ $\approx \lim_{x\to-\infty} \frac{4x}{-x} = \lim_{x\to-\infty} -4 = \boxed{-4}$

g. $\lim_{x\to\infty} \sin(x)$ $\boxed{\text{DNE}}$ because $\sin(x)$ oscillates between 1 and -1

h. $\lim_{x\to\infty} \frac{x^2}{e^{2x}}$ $= \boxed{0}$ because e^{2x} increases more rapidly than x^2

i. $\lim_{x\to\infty}(e^{8x} - 2e^{3x} + 5e^{2x} + 2e^{-2x})$ $= \lim_{x\to\infty} e^{8x}(1 - 2e^{-5x} + 5e^{-6x}) + \lim_{x\to\infty} 2e^{-2x}$

$= \infty(1-0+0) + 0 = \boxed{\infty}$

j. $\lim_{x\to\infty} \ln(7x^2 - 2x + 5)$

as $x \to \infty$, $7x^2 - 2x + 5 \to \infty$

so $\lim_{x\to\infty} \ln(7x^2 - 2x + 5) = \boxed{\infty}$

k. $\lim_{x\to-\infty} \ln\frac{1}{(x^2-x)}$ as $x \to -\infty$, $\frac{1}{x^2-x} \to 0^+$

so $\lim_{x\to-\infty} \ln \frac{1}{x^2-x} = \boxed{-\infty}$

9. Find the vertical and horizontal asymptotes for each function. Justify your answer using limits:

a. $f(x) = \frac{1}{x^2-4}$

$= \frac{1}{(x+2)(x-2)}$

$\lim_{x \to 2^-} f(x) = \frac{1}{\text{small} -} = -\infty$ $\quad \lim_{x \to 2^+} f(x) = \frac{1}{\text{small} +} = \infty$

$\lim_{x \to -2^-} f(x) = \frac{1}{\text{small} +} = \infty$ $\quad \lim_{x \to -2^+} f(x) = \frac{1}{\text{small} -} = -\infty$

$\boxed{\text{VA: } x = 2,\ x = -2}$

$\lim_{x \to \infty} \frac{1}{x^2-4} = \lim_{x \to \infty} \frac{\frac{1}{x^2}}{\frac{x^2}{x^2} - \frac{4}{x^2}} = \frac{0}{1-0} = 0$

$\boxed{\text{HA: } y = 0}$

$\lim_{x \to -\infty} \frac{1}{x^2-4} = \lim_{x \to -\infty} \frac{\frac{1}{x^2}}{\frac{x^2}{x^2} - \frac{4}{x^2}} = \frac{0}{1-0} = 0$

b. $f(x) = \frac{2x^2}{x+1}$

$\lim_{x \to -1^-} f(x) = \frac{2}{\text{small} -} = -\infty$ $\quad \lim_{x \to -1^+} f(x) = \frac{2}{\text{small} +} = \infty$

$\boxed{\text{VA: } x = -1}$

$\lim_{x \to \infty} \frac{2x^2}{x+1} = \lim_{x \to \infty} \frac{\frac{2x^2}{x^2}}{\frac{x}{x^2} + \frac{1}{x^2}} = \frac{2}{0+0} = \frac{2}{0}$ DNE

$\boxed{\text{No H.A.}}$

$\lim_{x \to -\infty} \frac{2x^2}{x+1} = \lim_{x \to -\infty} \frac{\frac{2x^2}{x^2}}{\frac{x}{x^2} + \frac{1}{x^2}} = \frac{2}{0+0} = \frac{2}{0}$ DNE

10. Find all numbers at which f is discontinuous:

a. $f(x) = \frac{3}{x^2+x-6}$ $= \frac{3}{(x+3)(x-2)}$

$\boxed{x \neq -3 \quad x \neq 2}$

b. $f(x) = \frac{x-4}{x^2-x-12}$ $= \frac{x-4}{(x-4)(x+3)}$ $\boxed{x \neq 4 \quad x \neq -3}$

c. $f(x) = \frac{x^2-9}{x-3}$ $= \frac{(x+3)(x-3)}{x-3}$ $\boxed{x \neq 3}$

d. $f(x) = \frac{5}{x^3-x^2}$ $= \frac{5}{x^2(x-1)}$ $\boxed{x \neq 0 \quad x \neq 1}$

e. $f(x) = \frac{\sqrt{x^2-9}\sqrt{25-x^2}}{x-4}$

$x^2 - 9 \geq 0$ $\quad 25 - x^2 \geq 0$ $\quad x \neq 4$

$x^2 \geq 9$ $\quad x^2 \leq 25$

$x \geq 3$ $\quad -5 \leq x \leq 5$

or

$x \leq -3$

$(-\infty, -5] \cup (-3, 3) \cup (4, 4) \cup (5, \infty)$

11. Verify the Intermediate Value Theorem for f on the stated interval $[a, b]$ by showing that if $f(a) \le w \le f(b)$, then $f(c) = w$ for some c in $[a, b]$.

a. $f(x) = x^3 + 1$ for $[-1,2]$

f continuous on $\mathbb{R}$, so continuous on $[-1, 2] \subset \mathbb{R}$

$f(-1) = 0$ $\quad$ $f(-1) \neq f(2)$

$f(2) = 9$

$\because$ by IVT $\forall w \in (0,9)\ \exists c \in (-1,2)$ such that $f(c) = w$

$c^3 + 1 = w$ $\quad$ we know $0 < w < 9$

$c^3 = w - 1$ $\quad$ $-1 < w - 1 < 8$

$c = \sqrt[3]{w-1}$ $\quad$ $-1 < \sqrt[3]{w-1} < 2$

$-1 < c < 2$

Q.E.D.

b. $f(x) = 2x - x^2$ for $[-2, -1]$

f continuous on $\mathbb{R}$, so continuous on $[-2, -1] \subset \mathbb{R}$

$f(-2) = -4 - (4) = -8$ $\quad$ $f(-2) \neq f(-1)$

$f(-1) = -2 - (1) = -3$

$\because$ by IVT, $\forall w \in (-8,-3)\ \exists c \in (-2,-1)$ such that $f(c) = w$

$2c - c^2 = w$

we know $-8 < w < -3$

$c^2 - 2c = -w$ $\quad$ $8 > -w > 3$

$c^2 - 2c + 1 = -w + 1$ $\quad$ $9 > -W + 1 > 4$

$(c-1)^2 = -w + 1$ $\quad$ $-3 < \sqrt{-w+1} < -2$

$c - 1 = \pm\sqrt{-w+1}$ $\quad$ $-2 < 1 - \sqrt{-w+1} < -1$

$c = 1 \pm \sqrt{-w+1}$ $\quad$ $-2 < c < -1$

Q.E.D.

12. Prove that $f(x) = 3x^4 + 2x^3 - 63x^2 - 102x - 40$ has a solution between 0 and 6.

$f(0) = 0 + 0 - 0 - 0 - 40 = -40$

$f(6) = 3(6^4) + 2(6^3) - 63(6^2) - 102(6) - 40 = 1400$

$\therefore$ by IVT $f(x)$ has a solution on $(0, 6)$

13. Determine A such that each function is continuous on $(-\infty, \infty)$:

a. $f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ A & \text{if } x = 2 \end{cases}$

$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x \to 2} (x+2)$

$= 4$

to be cont, $4 = f(2) = A$, so, $\boxed{A = 4}$

b. $g(x) = \begin{cases} \frac{\sin(4x)}{x} & \text{if } x \neq 0 \\ A & \text{if } x = 0 \end{cases}$

$\lim_{x \to 0} \frac{4\sin(4x)}{4x} = 4 \lim_{x \to 0} \frac{\sin(4x)}{4x} = 4 \cdot 1 = 4$

to be continuous

$f(0) = 4$

$f(0) = A = 4$

$\boxed{A = 4}$