1. $\lim_{x \to 2} 3x+1 = 7$

Given $\epsilon > 0$ choose $\delta = \frac{\epsilon}{3}$

suppose $0 < |x-2| < \delta$

Then $|3x+1-7| = |3x-6| = |3(x-2)| = 3|x-2| < 3\delta = 3\left(\frac{\epsilon}{3}\right) = \epsilon$

2. $\lim_{x \to -1} -5x-4 = 1$

Given $\epsilon > 0$ choose $\delta = \frac{\epsilon}{5}$

suppose $0 < |x+1| < \delta$

then $|-5x-4-1| = |-5x-5| = |-5(x+1)| = 5|x+1| < 5\delta = 5\left(\frac{\epsilon}{5}\right) = \epsilon$

3. $\lim_{x \to 2} x^2-5x+1 = -5$

Given $\epsilon > 0$ choose $\delta = \min\left\{2, \frac{\epsilon}{3}\right\}$

suppose $0 < |x-2| < \delta$

then $|x^2-5x+1-(-5)| = |x^2-5x+6| = |(x-2)(x-3)| = |x-2| \cdot |x-3|$

$< \delta \cdot |x-3| < 3\delta \leq 3\left(\frac{\epsilon}{3}\right) = \epsilon$

if $|x-2| < 2$

$-2 < x-2 < 2$

$-3 < x-3 < 1$

which means

$|x-3| < 3$

$\Downarrow$

$-3 < x-3 < 3$

4. $\lim_{x \to 3} x^2-5 = 4$

Given $\epsilon > 0$ choose $\delta = \min\left\{1, \frac{\epsilon}{7}\right\}$

suppose $0 < |x-3| < \delta$

then $|x^2-5-4| = |x^2-9| = |x+3| \cdot |x-3| < |x+3|\delta$

$< 7\delta \leq 7\left(\frac{\epsilon}{7}\right) = \epsilon$

if $|x-3| < 1$

$-1 < x-3 < 1$

$-5 < x+3 < 7$

$|x+3| < 7$

5. $\lim_{x\to 2} x^3+3x^2-4x = 12$

Given $\epsilon > 0$ choose $\delta = \min\left\{1, \frac{\epsilon}{30}\right\}$

Suppose $0 < |x-2| < \delta$

Then $|x^3+3x^2-4x-12| = |x^2(x+3)-4(x+3)| = |x+2|\cdot|x-2|\cdot|x+3|$

$< |x+2|\cdot|x+3|\cdot\delta < (5)(6)\delta = 30\delta \le 30\left(\frac{\epsilon}{30}\right) = \epsilon$

if $|x-2| < 1$

$-1 < x-2 < 1$

$3 < x+2 < 5$

$|x+2| < 5$

and

$4 < x+3 < 6$

$|x+3| < 6$

6. $\lim_{x\to -2} x^3-4 = -12$

Given $\epsilon > 0$ choose $\delta = \min\left\{1, \frac{\epsilon}{19}\right\}$

Suppose $0 < |x+2| < \delta$

Then $|x^3-4-(-12)| = |x^3+8| = |x+2|\cdot|x^2-2x+4|$

$< \delta\cdot|x^2-2x+4| \le \delta\cdot\{|x^2|+|-2x|+|4|\}$

$= \delta\cdot\{x^2+|-2|\cdot|x|+4\} = \delta\cdot\{x^2+2|x|+4\}$

$< \delta(9+2(3)+4) = 19\delta \le 19\left(\frac{\epsilon}{19}\right) = \epsilon$

if

$|x+2| < 1$

$-1 < x+2 < 1$

$-3 < x < -1$

$9 > x^2 > 1$

and

$|x| < 3$

7. $\lim_{x\to\infty} \frac{2x+3}{x} = 2$

Given $\epsilon > 0$ choose $N = \frac{3}{\epsilon}$

Suppose $x > N > 0$

Then $\left|\frac{2x+3}{x} - 2\right| = \left|\frac{2x+3-2x}{x}\right| = \left|\frac{3}{x}\right| = \frac{3}{x} < \frac{3}{N}$

$= \frac{3}{\frac{3}{\epsilon}} = \epsilon$

$x > N$

$\frac{1}{x} < \frac{1}{N}$

$\frac{3}{N} = \epsilon$

$N = \frac{3}{\epsilon}$

8. $\lim_{x\to -\infty} \frac{4x^2-1}{3x^2+5} = \frac{4}{3}$

Given $\epsilon > 0$ choose $N = -\sqrt{\frac{23}{\epsilon}}$

suppose $x < N < 0$

Then $\left|\frac{4x^2-1}{3x^2+5} - \frac{4}{3}\right| = \left|\frac{3(4x^2-1) - 4(3x^2+5)}{3(3x^2+5)}\right|$

$= \left|\frac{12x^2 - 3 - 12x^2 - 20}{9x^2+15}\right| = \left|\frac{-23}{9x^2+15}\right| = \frac{23}{9x^2+15}$

$< \frac{23}{9N^2+15} < \frac{23}{9N^2} < \frac{23}{N^2} = \frac{23}{\left(-\sqrt{\frac{23}{\epsilon}}\right)^2} = \epsilon$

for $x < N < 0$

$x^2 > N^2 > 0$

$9x^2+15 > 9N^2+15$

$\frac{1}{9x^2+15} < \frac{1}{9N^2+15}$

$\frac{23}{N^2} = \epsilon$

$N = \pm\sqrt{\frac{23}{\epsilon}}$

choose $-\sqrt{\frac{23}{\epsilon}}$

because $N < 0$

9. $\lim_{x\to\infty} \frac{3x-7}{5x^2+4x+1} = 0$

Given $\epsilon > 0$ choose $N = \max\left\{\frac{7}{3}, \frac{3}{5\epsilon}\right\}$

suppose $x > N > 0$

Then $\left|\frac{3x-7}{5x^2+4x+1}\right| = \frac{|3x-7|}{5x^2+4x+1} < \frac{|3x-7|}{5x^2} = \frac{3x-7}{5x^2}$ if $x > \frac{7}{3}$

$< \frac{3x}{5x^2} = \frac{3}{5x} < \frac{3}{5N} = \frac{3}{5\left(\frac{3}{5\epsilon}\right)} = \epsilon$

10. $\lim_{x\to 4^-} \frac{2x}{x-4} = -\infty$

Given $M < 0$ choose $\delta = \min\left\{1, -\frac{6}{M}\right\}$

suppose $0 < |x-4| < \delta$

Then $\frac{2x}{x-4} < \frac{2x}{-\delta} < \frac{6}{-\delta} \le \frac{6}{-\left(\frac{6}{M}\right)} = M$

if $x \to 4^-$, $x < 4$

$x-4 < 0$, $|x-4| = -(x-4)$

$-(x-4) < \delta$

$x-4 < -\delta$

$\frac{1}{x-4} < \frac{1}{-\delta}$

and if $|x-4| < 1$, $-1 < x-4 < 1$,

$3 < x < 5$, $\underline{6 < 2x < 10}$

11. $\lim_{x\to 2^+} \frac{2}{5x^2-20} = \infty$

$x \to 2^+$, $x > 2$

$x - 2 > 0$

Given $M > 0$ choose $\delta = \min\left\{1, \frac{2}{25M}\right\}$

so $x - 2 < \delta$

$\frac{1}{x-2} > \frac{1}{\delta}$

suppose $0 < |x-2| < \delta$

Then $\frac{2}{5x^2-20} = \frac{2}{5(x+2)(x-2)} > \frac{2}{5(x+2)\delta}$

if $x - 2 < 1$

$x + 2 < 5$

$\frac{1}{x+2} > \frac{1}{5}$

$> \frac{2}{5(5)\delta} = \frac{2}{25\delta} = \frac{2}{25\left(\frac{2}{25M}\right)} = M$

12. $\lim_{x\to 0^-} \frac{2x+1}{x^2} = \infty$

if $\delta = \frac{1}{4}$

Given $M > 0$ choose $\delta = \min\left\{\frac{1}{4}, \frac{1}{\sqrt{2M}}\right\}$

$|x| < \frac{1}{4}$

$x > -\frac{1}{4}$ becaus $x \to 0^-$

suppose $0 < |x| < \delta$

$2x > -\frac{1}{2}$

Then $\frac{2x+1}{x^2} > \frac{1}{2x^2} > \frac{1}{2\delta^2}$

$2x+1 > \frac{1}{2}$

and since $x < 0$

$= \frac{1}{2\left(\frac{1}{\sqrt{2M}}\right)^2} = M$

and $-x > 0$, $-x < \delta$

$(-x)^2 = x^2 < \delta^2$

$\frac{1}{x^2} > \frac{1}{\delta^2}$

$\frac{1}{2\delta^2} = M \to \delta = \frac{1}{\sqrt{2M}}$

13. $\lim_{x\to\infty} \frac{x^3-2}{x^2+5} = \infty$

if $x > \sqrt{5}$

Given $M > 0$ choose $N = \max\left\{\sqrt{5}, 2(M+1)\right\}$

$x^2 > 5$

suppose $x > N > 0$

$x^2 + x^2 > x^2 + 5$

Then $\frac{x^3-2}{x^2+5} > \frac{x^3-2}{2x^2} > \frac{x^3-2x^2}{2x^2}$

$2x^2 > x^2 + 5$

$\frac{1}{2x^2} < \frac{1}{x^2+5}$

$= \frac{x^2(x-2)}{2x^2} = \frac{x-2}{2} = \frac{x}{2} - 1$

and $x^2 > 5$

$x^3 - 2 > x^3 - 2x^2$

$> \frac{N}{2} - 1 = \frac{2(M+1)}{2} - 1 = M$

14. $\lim_{x \to \infty} \frac{3-x^4}{x^2+1} = -\infty$

Given $M<0$ choose $N = \sqrt{\sqrt{3} - M}$

Suppose $x > N > 0$

then $\frac{3-x^4}{x^2+1} < \frac{3-x^4}{x^2+\sqrt{3}} = \frac{-(x^4-3)}{x^2+\sqrt{3}} = \frac{-(x^2-\sqrt{3})(x^2+\sqrt{3})}{x^2+\sqrt{3}}$

$= \sqrt{3} - x^2 < \sqrt{3} - N^2$

$= \sqrt{3} - \left(\sqrt{\sqrt{3} - M}\right)^2 = M$

$x > N > 0$
$x^2 > N^2 > 0$
$-x^2 < -N^2 < 0$

15. $\lim_{x \to -\infty} \frac{-x^2}{x+1} = \infty$

Given $M>0$ choose $N = 1 - M$

Suppose $x < N < 0$

then $\frac{-x^2}{x+1} > \frac{-x^2+1}{x+1} = \frac{(1-x)(1+x)}{x+1} = 1 - x > 1 - N$

$= 1 - (1 - M) = M$

$x < N$
$-x > -N$
$1 - x > 1 - N$