1. $\lim_{x \to 3^+} \frac{4}{x-3} = \infty$ Given $M > 0$, choose $\delta = \frac{4}{M}$

suppose $0 < |x-3| < \delta$

and since $x > 3$, $x-3 > 0$ so $0 < x-3 < \delta$

and $\frac{1}{x-3} > \frac{1}{\delta}$

Then $\frac{4}{x-3} > \frac{4}{\delta} = \frac{4}{\left(\frac{4}{M}\right)} = M$

2. $\lim_{x \to 3^-} \frac{4}{x-3} = -\infty$ Given $M < 0$, choose $\delta = -\frac{4}{M}$

suppose $0 < |x-3| < \delta$

and since $x < 3$, $x-3 < 0$ so $0 < -(x-3) < \delta$

Then $\frac{4}{x-3} < \frac{4}{-\delta} = \frac{4}{-\left(-\frac{4}{M}\right)} = M$ and $x-3 > -\delta$

$\frac{1}{x-3} < -\frac{1}{\delta}$

3. $\lim_{x \to 3^+} \frac{4x}{x-3} = \infty$ Given $M > 0$, choose $\delta = \frac{12}{M}$

suppose $0 < |x-3| < \delta$

and since $x > 3$, $x-3 > 0$ so $0 < x-3 < \delta$

and $\frac{1}{x-3} > \frac{1}{\delta}$

Then $\frac{4x}{x-3} > \frac{4x}{\delta} > \frac{12}{\delta} = \frac{12}{\frac{12}{M}} = M$

4. $\lim_{x \to 2^+} \frac{x^2}{2x-4} = \infty$ Given $M > 0$, choose $\delta = \frac{2}{M}$

suppose $0 < |x-2| < \delta$

and since $x > 2$, $x-2 > 0$ so $0 < x-2 < \delta$

and $\frac{1}{x-2} > \frac{1}{\delta}$

Then $\frac{x^2}{2x-4} = \frac{x^2}{2(x-2)} > \frac{x^2}{2\delta}$

also $x > 2$
implies $x^2 > 4$

and $\frac{x^2}{2\delta} > \frac{4}{2\delta} = \frac{2}{\delta} = \frac{2}{\frac{2}{M}} = M$

5. $\lim_{x \to 1^-} \frac{5x^2}{6x-6} = -\infty$ Given $M<0$, choose $\delta = \min\left\{\frac{1}{2}, -\frac{5}{24M}\right\}$

Suppose $0 < |x-1| < \delta$ and since $x \to 1^-$, $x < 1$

so $x-1 < 0$

$|x-1| = -(x-1)$

so $-(x-1) < \delta$

$x-1 > -\delta$

$\frac{1}{x-1} < -\frac{1}{\delta}$

Then $\frac{5x^2}{6x-6} = \frac{5x^2}{6(x-1)} < \frac{5x^2}{6(-\delta)}$

$< \frac{5}{-24\delta} = \frac{5}{-24\left(-\frac{5}{24M}\right)} = M$

Since $x \to 1^-$, assume

$\frac{1}{2} < x < 1$

then $\frac{1}{4} < x^2 < 1$

and if $x^2 > \frac{1}{4}$

$\frac{5}{6}x^2 > \frac{5}{24}$

$\frac{5x^2}{6(-\delta)} < \frac{5}{24(-\delta)}$

6. $\lim_{x \to 5^+} \frac{x^3}{5-x} = -\infty$ Given $M < 0$, choose $\delta = \frac{-125}{M}$

suppose $0 < |x-5| < \delta$

since $x > 5$, $x - 5 > 0$ and $0 < x-5 < \delta$

$-(5-x) < \delta$

Then $\frac{x^3}{5-x} < \frac{x^3}{-\delta} < \frac{125}{-\delta} = \frac{125}{-\left(-\frac{125}{M}\right)} = M$

$5-x > -\delta$

$\frac{1}{5-x} < -\frac{1}{\delta}$

7. $\lim_{x \to \infty} \frac{1}{x+3} = 0$ Given $\epsilon > 0$ choose $N = \frac{1}{\epsilon}$

Suppose $x > N > 0$

since $x > 0$, $x+3 > 0$

and $\frac{1}{x+3} > 0$

Then $\left|\frac{1}{x+3} - 0\right| = \left|\frac{1}{x+3}\right| = \frac{1}{x+3}$

so $\left|\frac{1}{x+3}\right| = \frac{1}{x+3}$

and $\frac{1}{x+3} < \frac{1}{N+3} < \frac{1}{N} = \frac{1}{\left(\frac{1}{\epsilon}\right)} = \epsilon$

and $x > N \to x+3 > N+3$

and $\frac{1}{x+3} < \frac{1}{N+3}$

8. $\lim_{x \to -\infty} \frac{1}{x+3} = 0$ Given $\epsilon > 0$ Choose $N = -3 - \frac{1}{\epsilon}$

since $x < -3$, $x+3 < 0$

Suppose $x < N < -3$

and $\frac{1}{x+3} < 0$

Then $\left|\frac{1}{x+3} - 0\right| = \left|\frac{1}{x+3}\right| = -\frac{1}{x+3}$

so $\left|\frac{1}{x+3}\right| = -\frac{1}{x+3}$

and $-\frac{1}{x+3} < -\frac{1}{N+3} = -\frac{1}{\left(-3-\frac{1}{\epsilon}\right)+3} = \epsilon$

if $x < N$, $x+3 < N+3$

and $\frac{1}{x+3} > \frac{1}{N+3}$

if $\epsilon = -\frac{1}{N+3}$, $N+3 = -\frac{1}{\epsilon}$

$N = -3 - \frac{1}{\epsilon}$

and $-\frac{1}{x+3} < -\frac{1}{N+3}$

9. $\lim_{x\to\infty} \frac{3x}{x+5} = 3$ Given $\epsilon > 0$ choose $N = \frac{15}{\epsilon}$

suppose $x > N > 0$

then $\left|\frac{3x}{x+5} - 3\right| = \left|\frac{3x - 3(x+5)}{x+5}\right|$

$= \left|\frac{-15}{x+5}\right| = \frac{15}{x+5} < \frac{15}{N+5} < \frac{15}{N}$

and $\frac{15}{N} = \frac{15}{\left(\frac{15}{\epsilon}\right)} = \epsilon$

and $x > N$
$\to x+5 > N+5$
$\frac{1}{x+5} < \frac{1}{N+5}$

10. $\lim_{x\to-\infty} \frac{x^2-1}{3x^2+1} = \frac{1}{3}$ Given $\epsilon > 0$ choose $N = -\sqrt{\frac{4}{\epsilon}}$

suppose $x < N < 0$

Then $\left|\frac{x^2-1}{3x^2+1} - \frac{1}{3}\right| = \left|\frac{3(x^2-1) - (3x^2+1)}{3(3x^2+1)}\right| = \left|\frac{-4}{3(3x^2+1)}\right|$

$= \frac{4}{3(3x^2+1)} < \frac{4}{3(3N^2+1)} = \frac{4}{9N^2+3} < \frac{4}{9N^2} < \frac{4}{N^2}$

and $\frac{4}{N^2} = \frac{4}{\left(-\sqrt{\frac{4}{\epsilon}}\right)^2} = \epsilon$

if $x < N$
and $x < 0$
and $N < 0$
Then $x^2 > N^2$
and $3x^2 > 3N^2$
$3x^2+1 > 3N^2+1$
$\frac{1}{3x^2+1} < \frac{1}{3N^2+1}$

11. $\lim_{x \to \infty} \frac{2x}{x^2+1} = 0$ Given $\epsilon > 0$ Choose $N = \frac{2}{\epsilon}$

Suppose $x > N > 0$ → $x > N \to \frac{1}{x} < \frac{1}{N}$

Then $\left| \frac{2x}{x^2+1} - 0 \right| = \left| \frac{2x}{x^2+1} \right| = \frac{2x}{x^2+1}$

and $\frac{2x}{x^2+1} < \frac{2x}{x^2} = \frac{2}{x} < \frac{2}{N} = \frac{2}{\left(\frac{2}{\epsilon}\right)} = \epsilon$

12. $\lim_{x \to \infty} \frac{4x^2}{x+3} = \infty$ Given $M > 0$ choose $N = \max\{3, M\}$

suppose $x > N > 0$

Then $\frac{4x^2}{x+3} > \frac{4x^2}{2x} = 2x > x > N = M$

if $x > 3$
$x + x > x + 3$
$2x > x + 3$
$\frac{1}{2x} < \frac{1}{x+3}$

13. $\lim_{x \to -\infty} \frac{x^3+1}{x^2+1} = -\infty$ Given $M < 0$ choose $N = \min\{-1, 2M-1\}$

suppose $x < N < 0$

Then $\frac{x^3+1}{x^2+1} < \frac{x^3+1}{2x^2}$ if $x < -1$
so suppose
$x < N < -1$

since $x < N$
$x + 1 < N + 1$

and $\frac{x^3+1}{2x^2} < \frac{x^3+x^2}{2x^2} = \frac{x^2(x+1)}{2x^2}$

$= \frac{x+1}{2} < \frac{N+1}{2} = \frac{(2M-1)+1}{2} = M$

14. $\lim_{x \to \infty} \frac{2x^2+5}{4-x} = -\infty$ Given $M<0$ choose $N = -\frac{M}{2}$

suppose $x > N > 0$ and really $x > N > 4$ because of V.A. at $x=4$

then $\frac{2x^2+5}{4-x} < \frac{2x^2+5}{-x} < \frac{2x^2}{-x} = -2x$

and since $x > N$, $-2x < -2N$

then $-2x < -2N = -2\left(-\frac{M}{2}\right) = M$